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THE INVARIANT Π^0_α SEPARATION PRINCIPLE

BY

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ABSTRACT. We "invariantize" the classical theory of alternated unions to obtain new separation results in both invariant descriptive set theory and in infinitary logic. Application is made to the theory of definitions of countable models.

0. Introduction. In this paper we will be concerned with some results related to the theorem: *Disjoint G_δ sets in a complete metric space can be separated by an alternated union of closed sets*. Before summarizing the contents of the paper it will be helpful to recall some classical definitions and results.

Let X be an arbitrary set or class.

Suppose Γ_1, Γ_2 are two subclasses of $\mathcal{P}(X)$ such that $\Gamma_2 \subseteq \Gamma_1$ and Γ_2 is closed under complementation. Γ_1 *has the strong separation property with respect to* Γ_2 provided that for $A_0, A_1 \in \Gamma_1$, if $A_0 \cap A_1 = \varnothing$, then there exists $B \in \Gamma_2$ such that $A_0 \subseteq B \subseteq \sim A_1$ (i.e. B *separates* A_0 from A_1). An equivalent condition is that Γ_1 has the first separation property and $\Gamma_1 \cap \check{\Gamma}_1 = \Gamma_2$ (cf. Addison [1] for a discussion of this phenomenon). $\check{\Gamma}_1 = \{\sim A: A \in \Gamma_1\}$.

ON is the class of all ordinals. Let $\gamma \in \mathrm{ON}$ and suppose $C = \langle C_\beta: \beta \leqslant \gamma\rangle$ is a sequence of subclasses of X. C is *decreasing* if $C_\beta \subseteq C_{\beta'}$ whenever $\beta' < \beta \leqslant \gamma$. C is *continuous* if $C_\lambda = \bigcap_{\beta<\lambda} C_\beta$ whenever $\lambda \leqslant \gamma$ is a limit ordinal. $e(\gamma) = \{\beta \in \gamma: \beta \text{ is even}\}$. $D(C) = \bigcup\{C_\beta \sim C_{\beta+1}: \beta \in e(\gamma)\}$ is the alternated union of C.

Let $\Gamma \subseteq \mathcal{P}(X)$. C is *suitable* for $\mathfrak{D}_\gamma(\Gamma)$ if $C \in {}^{\gamma+1}\Gamma$, C is decreasing and continuous, $C_0 = X$ and $C_\gamma = \varnothing$. We define $\mathfrak{D}_\gamma(\Gamma) = \{D(C): C \text{ is suitable for } \mathfrak{D}_\gamma(\Gamma)\}$, $\mathfrak{D}_{(\gamma)}(\Gamma) = \bigcup\{\mathfrak{D}_\nu(\Gamma): \nu < \gamma\}$, $\mathfrak{D}_{(\infty)}(\Gamma) = \bigcup\{\mathfrak{D}_\nu(\Gamma): \gamma \in \mathrm{ON}\}$. $\mathfrak{D}_{(\omega_1)}(\Gamma)$ is the collection of countable alternated unions over Γ.

The important feature of alternated unions is their behavior under complementation. If C is suitable for $\mathfrak{D}_\gamma(\mathcal{P}(X))$, then it is easily seen (cf. Kuratowski [9]) that

$$\sim D(C) = \bigcup \{C_{\beta-1} \sim C_\beta: \beta \in e(\gamma), \beta \text{ a successor}\}. \tag{1}$$

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It follows that if Γ_1 is a class which includes $\Gamma \cup \check{\Gamma}$ and is closed under finite intersections and countable unions, then $\mathfrak{D}_{(\omega_1)}(\Gamma) \subseteq \Gamma_1 \cap \check{\Gamma}_1$.

Now suppose X is a topological space. $\mathbf{\Pi}^0_\alpha$ and $\mathbf{\Sigma}^0_\alpha$ are, respectively, the αth multiplicative and additive levels of the Borel hierarchy on X ($\mathbf{\Pi}^0_1$ is the collection of closed sets, $\mathbf{\Pi}^0_2 = G_\delta$, etc.). We further specify

$$\mathbf{\Delta}^0_\alpha = \mathbf{\Pi}^0_\alpha \cap \mathbf{\Sigma}^0_\alpha, \qquad \mathbf{\Pi}^0_{(\alpha)} = \bigcup \{\mathbf{\Pi}^0_\beta : \beta < \alpha\}.$$

When we wish to emphasize the dependence on X, we write $\mathbf{\Pi}^0_\alpha(X)$, etc.

$\mathfrak{D}_{(\infty)}(\mathbf{\Pi}^0_1)$ was known classically as the collection of *resolvable* sets. A result of Montgomery (cf. [9, §34]) states that $\mathfrak{D}_{(\infty)}(\mathbf{\Pi}^0_1) \subseteq \mathbf{\Delta}^0_2$ provided X is metrizable (for separable X this is obvious). The basic $\mathbf{\Pi}^0_2$ separation theorem (2) is essentially due to Hausdorff (cf. Kuratowski [9] or Addison [2]).

(2) *Assume X is completely metrizable. Then $\mathbf{\Pi}^0_2$ has the strong separation property with respect to $\mathfrak{D}_{(\infty)}(\mathbf{\Pi}^0_1)$.*

When X is Polish (separable, completely metrizable), (2) can be extended to all higher levels of the Borel hierarchy.

(3) *Assume X is Polish, $\omega_1 > \alpha > 1$. Then $\mathbf{\Pi}^0_\alpha$ has the strong separation property with respect to $\mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}^0_{(\alpha)})$.*

(3) is usually proved only for successor α (cf. [9, §37]). For $\alpha = \lambda$ a limit ordinal, the situation is simpler. One easily shows that $\mathbf{\Delta}^0_\lambda = \mathfrak{D}_\omega(\mathbf{\Pi}^0_{(\lambda)})$, and (3) follows from the fact that $\mathbf{\Sigma}^0_\lambda$ has the reduction property [9, §30].

Given an equivalence relation on X, it is natural to ask whether (2) and (3) hold in "invariant" form. In §1 we will answer this question affirmatively for suitable E. If E is an open (lower semicontinuous) equivalence, then the collection of E-invariant G_δ sets has the strong separation property with respect to the collection of alternated unions of E-invariant closed sets. If E is induced by a "Polish action" (§1), then an analogous invariant version of (3) holds.

In §2 we combine this invariant result with a definability theorem of Vaught [15] to obtain an analogous fact in logic.

(4) *If ρ is countable, $\omega_1 > \alpha > 1$, then $\mathbf{\Pi}'^0_\alpha(V_\rho)$ has the strong separation property with respect to $\mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_{(\alpha)}(V_\rho))$.*

Here $\mathbf{\Pi}'^0_\alpha(V_\rho)$ is the αth multiplicative level in the Borel′ hierarchy (§2) on the $L_{\omega_1\omega}$-elementary classes. For example, $\mathbf{\Pi}'^0_2$ classes have the form $\mathrm{Mod}(\bigwedge_n \forall \check{x} \bigvee_m \exists \check{y} \theta_{mn})$ where each θ_{mn} is finitary, quantifier-free, $n, m \in \omega$.

An interesting aspect of (4) is its relation to the well-known $\forall^0_n$-separation theorem of first-order logic (cf. [13, p. 97]). This latter result was conjectured by Addison by analogy with (3), and was established by Shoenfield, cf. [1]. It can be derived from (4) and a general approximation theorem due to J. Keisler–see Remark II, §2 below. Thus, our proof of (4) as a consequence of (3) gives a kind of "causal explanation" for Addison's analogy.

In §3 we use the invariant Π_α^0 separation theorem to derive several results on the complexity of $L_{\omega_1\omega}$ definitions of isomorphism types. The following is typical:

(5) *If a complete $L_{\omega\omega}$ theory T has a countable model $\mathfrak{A}$ such that the isomorphism type of $\mathfrak{A}$ is $\Sigma_2'^0$-over-$L_{\omega\omega}$, then T is ω-categorical.*

$\Sigma_2'^0$-over-$L_{\omega\omega}$ classes have the form $\mathrm{Mod}(\bigvee_n \exists \check{x} \bigwedge_m \forall \check{y} \theta_{nm}(\check{x}, \check{y}))$ each $\theta_{nm} \in L_{\omega\omega}$. These results can also be stated in terms of topological complexity in a natural space of countable models. This space has been studied by several authors, most recently by Benda [5]. When stated topologically, both (5) and 3.5(ii) improve one of the results of [5]. We also indicate how the invariant separation theorem can be used in ordinary descriptive set theory by giving a new proof of a classical definability theorem of Baire.

§4 is almost independent of the previous sections and deals with the problem of effectiveness. We prove "admissible" versions of (3) and (4) showing that they remain valid when we restrict our attention to Π_α^0 sets and $\Pi_\alpha'^0$ classes with "names" in any admissible set, provided α is a successor ordinal.

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1. The invariant Π_α^0 separation theorem. Assume X is a topological space and E is an equivalence relation on X. For $B \subseteq X$, define

$$B^{-E} = B^- = \{x: (\forall y)(yEx \Rightarrow y \in B)\},$$
$$B^+ = \sim(\sim B)^- = \{x: (\exists y)(yEx \,\&\, y \in B)\}.$$

B is *invariant* if $B = B^-$. B^- and B^+ are respectively the largest invariant subset of B and the smallest invariant superset of B. If $B^\#$ is an invariant set such that $B^- \subseteq B^\# \subseteq B^+$, we say that $B^\#$ is an *invariantization* of B.

Note that

(6) If $B^\#$ is an invariantization of B and B separates a pair of disjoint invariant sets, then so does $B^\#$.

Thus, the invariant Π_2^0 separation problem is connected with the Δ_2^0 invariantization problem: "Given $B \in \Delta_2^0$, find a Δ_2^0 invariantization for B." In view of (2), we can solve the Δ_2^0 invariantization problem for X, when X is completely metrizable, by solving each $\mathfrak{D}_\gamma(\Pi_1^0)$ invariantization problem.

Given $\Gamma \subseteq \mathcal{P}(X)$, inv($\Gamma$) denotes $\{B \in \Gamma: B \text{ is invariant}\}$.

Given $C \in {}^\gamma(\mathcal{P}(X))$, let $C^\ominus = \langle C_\beta^-: \beta < \gamma \rangle$.

LEMMA 1.1. *Assume C is suitable for $\mathfrak{D}_\gamma(\mathcal{P}(X))$. Then $C^\ominus$ is suitable for*

$\mathfrak{D}_\gamma(\mathrm{inv}(\mathcal{P}(X)))$ *and* $D(C^\ominus)$ *is an invariantization of* $D(C)$.

PROOF. First note that for each $\beta < \gamma$,

$$C_\beta^- \sim C_{\beta+1}^- = C_\beta^- \cap (\sim C_{\beta+1})^+ \subseteq (C_\beta \sim C_{\beta+1})^+.$$

It follows that

$$D(C^\ominus) = \bigcup \{C_\beta^- \sim C_{\beta+1}^- : \beta \in e(\gamma)\} \subseteq \bigcup \{(C_\beta \sim C_{\beta+1})^+ : \beta \in e(\gamma)\}$$
$$= \left(\bigcup \{C_\beta \sim C_{\beta+1} : \beta \in e(\gamma)\}\right)^+ = (D(C))^+.$$

A similar calculation based on (1) shows that $\sim D(C^\ominus) \subseteq (\sim D(C))^+ = \sim(D(C)^-)$. $D(C^\ominus)$ is clearly invariant, hence it is an invariantization of $D(C)$. Since the transform $C_\beta \mapsto C_\beta^-$ preserves inclusions and commutes with intersections, $C^\ominus$ is suitable for $\mathfrak{D}_\gamma(\mathrm{inv}(\mathcal{P}(X)))$. $\square$

E is *open* (*lower semicontinuous*) if C^- is closed whenever C is closed. An equivalent condition is that the canonical map $X \to X/E$ be an open mapping. For example, every equivalence which is induced by a group of autohomeomorphisms of X is open.

THEOREM 1.2. *Assume X is a topological space and E is open equivalence on X.*

(a) *For every* $\gamma \in \mathrm{ON}$, $\mathrm{inv}(\mathfrak{D}_\gamma(\mathbf{\Pi}_1^0)) = \mathfrak{D}_\gamma(\mathrm{inv}(\mathbf{\Pi}_1^0))$.

(b) *If X is completely metrizable, then* $\mathrm{inv}(\mathbf{\Pi}_2^0)$ *has the strong separation property with respect to* $\mathfrak{D}_{(\infty)}(\mathrm{inv}(\mathbf{\Pi}_1^0))$.

PROOF. If B is closed and E is open, then $B^- \in \mathrm{inv}(\mathbf{\Pi}_1^0)$. Thus, $C^\ominus \in^{\gamma+1}(\mathrm{inv}(\mathbf{\Pi}_1^0))$ when $C \in^{\gamma+1}(\mathbf{\Pi}_1^0)$. If $D(C)$ is invariant, then, by (6) and 1.1, $D(C) = D(C^\ominus)$. This proves (a). Now suppose X is completely metrizable, $A_0, A_1 \in \mathrm{inv}(\mathbf{\Pi}_2^0)$. Applying (2), let C be suitable for $\mathfrak{D}_{(\infty)}(\mathbf{\Pi}_1^0)$, $A_0 \subseteq D(C) \subseteq \sim A_1$. Then, again by (6) and 1.1, $D(C^\ominus) \in \mathfrak{D}_{(\infty)}(\mathrm{inv}(\mathbf{\Pi}_1^0))$ and $A_0 \subseteq D(C^\ominus) \subseteq \sim A_1$. $\square$

Before giving a similar invariant version of (3), we review some of the definitions and results of Vaught [15]. Let G be a topological space. Recall that a subset of G is meager (first category) if it is a countable union of nowhere dense sets. G is a Baire space if no nonempty open subset of G is meager.

Assume for the remainder of §1 that G is a Baire space, X and X' are sets (possibly with additional structure), and J is a function on $G \times X$ to X'. For $B \subseteq X'$, $x \in X$, define $B^x = \{g: J(g,x) \in B\}$, $B^{+J} = \{x: B^x \neq \varnothing\}$, $B^{-J} = \sim(\sim B)^{+J}$. Further define $B^* = \{x: B^x \text{ is comeager}\}$, $B^\Delta = \sim(\sim B)^*$. Since G is a Baire space, it is apparent that for $B \subseteq X'$,

$$B^{-J} \subseteq B^* \subseteq B^\Delta \subseteq B^{+J}. \tag{7}$$

For $B_i \subseteq X'$, $i \in \omega$, we also have

$$\Big(\bigcap_{i\in\omega} B_i\Big)^* = \bigcap_{i\in\omega} B_i^*, \qquad \Big(\bigcup_{i\in\omega} B_i\Big)^\Delta = \bigcup_{i\in\omega} B_i^\Delta. \tag{8}$$

For $g \in G$ define $J^g\colon X \to X'$ by setting $J^g(x) = J(g, x)$. If G is a group, $X = X'$, and the map $g \mapsto J^g$ is a homomorphism on G to the group of permutations of X, then $\mathcal{J} = (G, X, J)$ is an *action*. If, moreover, G is a (Baire) topological group with a countable base, X is a topological space and J is continuous in each variable separately, then $\mathcal{J}$ is a *special action*. An important particular case is that of a *Polish action*– G and X are Polish spaces and J is fully continuous.

Let (G, X, J) be an action and let $E_J = \{(x, x')\colon (\exists g)(J(g, x) = x')\}$. Then E_J is an equivalence on X and, for $B \subseteq X$, $B^{-J} = B^{-E_J}$. It follows from (7) and the homogeneity of topological groups that

(9) If G is a Baire topological group, (G, X, J) is an action, and $B \subseteq X$, then both B^* and B^Δ are invariantizations of B (with respect to E_J).

In **[15]** Vaught solved each Π^0_α invariantization problem for special actions by proving:

(10) Assume (G, X, J) is a special action, $1 \leqslant \alpha < \omega_1$, $B \in \Pi^0_\alpha$. Then $B^* \in \Pi^0_\alpha$.

Note that (10) does not directly solve the Δ^0_α invariantization problem. If $B \in \Delta^0_\alpha$, then $B^* \in \Pi^0_\alpha$, $B^\Delta \in \Sigma^0_\alpha$, but neither B^* nor B^Δ is necessarily a member of Δ^0_α. As in 1.1 and 1.2, we will solve the Δ^0_α invariantization problems for special actions on Polish spaces by considering alternated unions.

Return to the basic hypothesis on G, X, X', J. For $\gamma \in \omega_1$, $C \in {}^\gamma\mathcal{P}(X')$ define $C^{\circledast} = \langle C^*_\beta\colon \beta < \gamma\rangle$.

LEMMA 1.3. *Assume $\gamma \in \omega_1$ and C is suitable for $\mathfrak{D}_\gamma(\mathcal{P}(X'))$. Then $C^{\circledast}$ is suitable for $\mathfrak{D}_\gamma(\mathcal{P}(X))$ and $(D(C))^* \subseteq D(C^{\circledast}) \subseteq (D(C))^\Delta$.*

PROOF. Since the intersection of a comeager subset of G with a nonmeager set is nonmeager, we have for each $\beta < \gamma$,

$$C^*_\beta \sim C^*_{\beta+1} = C^*_\beta \cap (\sim C_{\beta+1})^\Delta \subseteq (C_\beta \sim C_{\beta+1})^\Delta.$$

Since the transform $B \mapsto B^\Delta$ commutes with countable unions and the transform $B \mapsto B^*$ commutes with countable intersections and preserves inclusions, we may substitute "$*$" for "$-$", "Δ" for "$+$" in the proof of 1.1 to obtain a proof of 1.3. □

THEOREM 1.4. *Assume that (G, X, J) is a special action, $1 < \alpha < \omega_1, \gamma < \omega_1$, $E = E_J$.*

(a) *If C is suitable for* $\mathfrak{D}_\gamma(\mathbf{\Pi}^0_{(\alpha)})$, *then* $C^{\circledast}$ *is suitable for* $\mathfrak{D}_\gamma(\mathrm{inv}(\mathbf{\Pi}^0_{(\alpha)}))$ *and* $D(C^{\circledast})$ *is an invariantization of* $D(C)$.

(b) $\mathrm{inv}(\mathfrak{D}_\gamma(\mathbf{\Pi}^0_{(\alpha)})) = \mathfrak{D}_\gamma(\mathrm{inv}(\mathbf{\Pi}^0_{(\alpha)}))$.

(c) *If X is Polish, then* $\mathrm{inv}(\mathbf{\Pi}^0_\alpha)$ *has the strong separation property with respect to* $\mathfrak{D}_{(\omega_1)}(\mathrm{inv}(\mathbf{\Pi}^0_{(\alpha)}))$.

PROOF. The proof is entirely parallel to that of 1.2.

To establish (a), let $C = \langle C_\beta: \beta < \gamma\rangle$. By (10) each C^*_β is invariant $\mathbf{\Pi}^0_{(\alpha)}$ so $D(C^{\circledast})$ is suitable for $\mathfrak{D}_\gamma(\mathrm{inv}(\mathbf{\Pi}^0_{(\alpha)}))$. By (9) and 1.3, $D(C^{\circledast})$ is an invariantization of $D(C)$.

(b) is immediate from (a) since an invariant set is its own only invariantization.

To establish (c), let A_0, A_1 be disjoint invariant $\mathbf{\Pi}^0_\alpha$ sets. Applying (3) let C be suitable for $\mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}^0_{(\alpha)})$, $A_0 \subseteq D(C) \subseteq \sim A_1$. By (6) and (a), $D(C^{\circledast}) \in \mathfrak{D}_{(\omega_1)}(\mathrm{inv}(\mathbf{\Pi}^0_{(\alpha)}))$ and $A_0 \subseteq D(C^{\circledast}) \subseteq \sim A_1$. □

2. The $\mathbf{\Pi}'^0_\alpha$ separation theorem. Let ρ be a countable similarity type. Without loss of generality (see Remark I below) we assume that ρ is simply a set of relation symbols. For $\mathbf{P} \in \rho$, let $n(\mathbf{P})$ be the arity of $\mathbf{P}$. V_ρ is the class of all ρ-structures. Given a collection Ω of sentences of $L_{\omega_1\omega}(\rho)$ and a class $K \subseteq V_\rho$, let $\Omega(K) = \{\mathrm{Mod}(\sigma) \cap K: \sigma \in \Omega\}$. V^∞_ρ is the class of infinite ρ-structures and for $\sigma \in L_{\omega_1\omega}(\rho)$, $\mathrm{Mod}^\infty(\sigma) = \mathrm{Mod}(\sigma) \cap V^\infty_\rho$. A *fragment* of $L_{\omega_1\omega}(\rho)$ is a subset of $L_{\omega_1\omega}$ which is closed under negation, quantification, finite conjunction and disjunction, passage to subformulas, and substitution of variables.

The *canonical logic space of type* ρ is the topological product space $X_\rho = \Pi_{\mathbf{P}\in\rho} 2^{n(\mathbf{P})}$. We identify $S \in X_\rho$ with (ω, S) to view X_ρ as the set of ρ-structures having universe ω. Given a sentence $\sigma \in L_{\omega_1\omega}(\rho)$, we set $[\sigma] = \mathrm{Mod}(\sigma) \cap X_\rho$. The *canonical logic action of type* ρ is $\mathcal{J}_\rho = (X_\rho, \omega!, J_\rho)$, where $\omega!$ is the group of permutations of the set ω given the relative topology as a subspace of ω^ω, and J_ρ is defined by setting

$$J_\rho(g, S)_{\mathbf{P}}(i_1, \ldots, i_{n(\mathbf{P})}) = S\big(g^{-1}(i_1), \ldots, g^{-1}(i_{n(\mathbf{P})})\big).$$

Thus, $J_\rho(g, S) = gS$ is the usual isomorph of S under g and $E_{J_\rho} = I_\rho$ is the usual isomorphism relation between ρ-structures. It is easily seen that X_ρ is a Polish space and $\mathcal{J}_\rho$ is a Polish action (cf. [15]). All action-theoretic terms in this section will refer to this action.

A basic′ or $\mathbf{\Pi}'^0_0$ formula of type ρ is a finite conjunction of atomic formulas and negations of atomic formulas. The Borel′ hierarchy of formulas of $L_{\omega_1\omega}(\rho)$ is then defined for $1 \leq \alpha < \omega_1$ by the recursive conditions:

$\mathbf{\Pi}'^0_\alpha = \{\neg\phi: \phi \in \mathbf{\Sigma}'^0_\alpha\}$;

$\mathbf{\Pi}'^0_{(\alpha)} = \bigcup_{\beta<\alpha} \mathbf{\Pi}'^0_\beta$;

$\mathbf{\Sigma}'^0_\alpha = \{\bigvee\Theta: \Theta$ is countable and each $\theta \in \Theta$ is of the form $\exists v_1 \cdots \exists v_k \psi$, where $k \in \omega$, each v_i is a variable, and $\psi \in \mathbf{\Pi}'^0_{(\alpha)}\}$.

In [15] Vaught proved

(11) Let $\alpha \geqslant 1$. If $B \in \mathbf{\Pi}_\alpha^0(X_\rho)$ then $B^* \in \mathbf{\Pi}'^0_\alpha(X_\rho)$.

Combining this with (9) he obtained

(12) Let $\alpha \geqslant 1$ and suppose X is an invariant subspace of X_ρ. Then $\operatorname{inv}(\mathbf{\Pi}_\alpha^0(X)) = \mathbf{\Pi}'^0_\alpha(X)$.

The main result of this section is the $\mathbf{\Pi}'^0_\alpha$ separation theorem: *Disjoint* $\mathbf{\Pi}'^0_\alpha$ *classes can be separated by a countable alternated union of* $\mathbf{\Pi}'^0_{(\alpha)}$ *classes*, $\alpha \geqslant 2$.

Over infinite models this result is an immediate consequence of (12) and Theorem 1.4. If $\alpha \geqslant 2$ and σ, θ are $\mathbf{\Pi}'^0_\alpha$ sentences such that $\operatorname{Mod}^\infty(\sigma) \cap \operatorname{Mod}^\infty(\theta) = \varnothing$, then $[\sigma]$, $[\theta]$ are disjoint invariant $\mathbf{\Pi}_\alpha^0$ subsets of X_ρ. By 1.4, $[\sigma] \subseteq D(C) \subseteq [\neg\theta]$ for some countable sequence $C = \langle C_\beta : \beta < \gamma \rangle$ suitable for $\mathfrak{D}_\gamma(\operatorname{inv} \mathbf{\Pi}^0_{(\alpha)})$. By (12), each C_β is $[\phi_\beta]$ for some $\phi_\beta \in \mathbf{\Pi}'^0_{(\alpha)}$. By the infinitary Löwenheim-Skolem theorem, $\langle \operatorname{Mod}^\infty(\phi_\beta) : \beta < \gamma \rangle = \Phi$ is decreasing and continuous and $\operatorname{Mod}^\infty(\sigma) \subseteq D(\Phi) \subseteq \operatorname{Mod}^\infty(\neg\theta)$.

Since ρ is assumed to be countable, every collection of finite ρ-structures is $\mathbf{\Sigma}'^0_2(V_\rho)$. Using this fact, the full $\mathbf{\Pi}'^0_\alpha$ separation theorem for $\alpha > 2$ is easily obtained from the corresponding result over infinite models. In dealing with $\mathbf{\Pi}'^0_2$ and with problems of effectiveness however, this *ad hoc* approach to finite models breaks down. We will solve the problem by considering a variant of the usual logic space and proving a definability theorem analogous to (12).

The (familiar) trick is to treat equality as a nonlogical symbol so that an infinite set of natural numbers can represent a single element of a finite structure.

Let $\approx$ be a binary relation symbol which is not in ρ and let $\bar{\rho} = \rho \cup \{\approx\}$. Let $\overline{X}_\rho \subseteq X_{\bar{\rho}}$ be the collection of all $(S, \sim)$ such that $\sim$ is a congruence on ω for each relation in S and each congruence class is infinite. Since each equality axiom is $\mathbf{\Pi}'^0_1$, $\overline{X}_\rho$ is $\mathbf{\Pi}'^0_2$ in $X_{\bar{\rho}}$. Given $(S, \sim) \in \overline{X}_\rho$, the natural quotient structure $(S, \sim)/\sim$ is a ρ-structure and it is apparent that every finite or infinite countable ρ-structure can be obtained as such a quotient.

Given $X \subseteq X_\rho$, $n \in \omega$, let $X^{(n)} = \{(S, i_1, \ldots, i_n) : S \in X \;\&\; i_1, \ldots, i_n$ are distinct natural numbers$\}$. An n-formula is a formula with free variables included in $\{\mathrm{v}_0, \ldots, \mathrm{v}_{n-1}\}$. If $\theta \in L_{\omega_1\omega}(\rho)$ is an n-formula, define

$$[\theta^{(n)}] = \operatorname{Mod}(\theta) \cap X_\rho^{(n)} = \{(S, \check{i}) \in X_\rho^{(n)} : (\omega, S, \check{i}) \models \theta\}.$$

Given $\phi \in L_{\omega_1\omega}(\rho)$ let $\bar{\phi} = \phi(\underset{\approx}{=})$ be the result of substituting $\approx$ for the equality symbol $=$ throughout ϕ. Clearly $\bar{\phi}$ has the same position in the Borel$'$ hierarchy on $\bar{\rho}$ that ϕ has in the hierarchy on ρ. Furthermore, if ϕ is an n-formula, then

$$[\bar{\phi}^{(n)}] \cap \overline{X}_\rho^{(n)} = \{(S, \sim, i_1, \ldots, i_n) : ((\omega, S \sim)/\sim, [i_1], \ldots, [i_n]) \models \phi\}.$$

Here, $[i]$ is the equivalence class of i under $\sim$.

$[\bar{\phi}^{(n)}] \cap \bar{X}_\rho^{(n)}$ will be denoted $(\!|\phi^{(n)}|\!)$. As usual we drop the superscript when $n = 0$. Given any class Γ of ρ-formulas, we let $\bar{\Gamma} = \{\bar{\phi}: \phi \in \Gamma\}$.

Since each $\sim$ is a congruence, any isomorphism between structures $(S, \sim)$, $(S', \sim') \in \bar{X}_\rho$ induces an isomorphism between the corresponding quotients. It follows that each class $(\!|\phi|\!)$ is an $I_{\bar{\rho}}$-invariant subset of $\bar{X}_\rho$, and we have for each $\alpha \geqslant 1$,

$$\overline{\Pi'^0_\alpha}\left(\bar{X}_\rho\right) \subseteq \operatorname{inv}\left(\Pi^0_\alpha\left(\bar{X}_\rho\right)\right).$$

Since all congruence classes have the same cardinality, any isomorphism between quotient structures $(S, \sim)/\sim$ and $(S', \sim')/\sim'$ can be lifted to an isomorphism between the structures $(S, \sim)$ and $(S', \sim')$. Thus, $I_{\bar{\rho}}$ is the natural equivalence on $\bar{X}_\rho$ to study for applications to logic.

With a minor modification of the proof, Vaught's main definability results go over to the new context:

PROPOSITION 2.1. *Let* $\alpha \geqslant 1$.
(a) *If* $B \in \Pi^0_\alpha(\bar{X}_\rho)$ *then* $B^* \in \overline{\Pi'^0_\alpha}(\bar{X}_\rho)$.
(b) *If* X *is any invariant subspace of* $\bar{X}_\rho$, *then* $\operatorname{inv}(\Pi^0_\alpha(X)) = \overline{\Pi'^0_\alpha}(X)$.

PROOF. Before proceeding with the proof of 2.1 note that (12) is easily derived from 2.1(b) by considering the invariant subspace $X = \{(S, \sim)$: $(S \sim)/\sim$ is infinite$\}$. The obstacle to a similar derivation of 2.1(b) from (12) applied to the invariant subspace $\bar{X}_\rho \subseteq X_{\bar{\rho}}$ is that (12) is proved for logic *with equality*. We will see that in our special case, this "extra" logical equality symbol can be eliminated.

Given a formula ϕ, let $(\exists^{\neq} \mathbf{v}_m \cdots \mathbf{v}_{n-1})(\phi)$ abbreviate $(\exists \mathbf{v}_m \cdots \mathbf{v}_{n-1})(\phi \wedge \bigwedge_{i<j<n} \mathbf{v}_i \neq \mathbf{v}_j)$. The key remark which enables us to modify the argument of [15] is

(14) If ψ is an m-formula of $L_{\omega_1\omega}(\bar{\rho})$ such that the symbol $=$ does not occur in ψ, and $n \leqslant m$, then

$$\left[(\exists^{\neq} \mathbf{v}_n \cdots \mathbf{v}_{m-1})(\psi)^{(n)}\right] \cap \bar{X}_\rho^{(n)} = \left[(\exists \mathbf{v}_n \cdots \mathbf{v}_{m-1})(\psi)^{(n)}\right] \cap \bar{X}_\rho^{(n)}.$$

The inclusion from left to right in (14) is trivial. For the reverse inclusion let $(S, \sim, i_0, \ldots, i_{n-1}) \in \bar{X}_\rho^{(n)}$ and suppose $(\omega, S, \sim, i_0, \ldots, i_{m-1}) \vDash \psi$. Since each congruence class is infinite, there exist numbers $i'_n, \ldots, i'_{m-1}$ such that $i_0, \ldots, i_{n-1}, i'_n, \ldots, i'_{m-1}$ are distinct and $i_j \sim i'_j$ for $j = n, \ldots, m-1$. Since $\sim$ is a congruence,

$$(\omega, S, \sim, i_0, \ldots, i_{n-1}, i'_n, \ldots, i'_{m-1}) \vDash \psi$$

and

$$(S, \sim, i_0, \ldots, i_{n-1}) \in \left[(\exists^{\neq} \mathbf{v}_n \cdots \mathbf{v}_{m-1})(\psi)^{(n)}\right]$$

as required, establishing (1.4).

Now we sketch a proof of (a). As in [15] it is necessary to prove a somewhat stronger result. For $B \subseteq X_{\bar{\rho}}$, let $B^{(*n)}, B^{(\Delta n)} \subseteq X^{(n)}_{\bar{\rho}}$ be defined as in [15]. (Our $X^{(n)}_{\bar{\rho}}$ differs slightly from that in [15], but Vaught's definition of $B^{(*n)}$, $B^{(\Delta n)}$ makes sense in our context and yields the same sets.) Using (14) we can modify the proof of [15] 3.1 to establish by induction on α:

(15) Let $B \in \Pi^0_\alpha(X_{\bar{\rho}})$ (respectively, $\Sigma^0_\alpha(X_{\bar{\rho}})$). Then for each $n \in \omega$ there is an n-formula $\psi \in \Pi'^0_\alpha(\bar{\rho})$ (resp. $\Sigma'^0_\alpha(\bar{\rho})$) *which does not contain the equality symbol*, such that

$$B^{(*n)} \cap \bar{X}^{(n)}_\rho = [\psi^{(n)}] \cap \bar{X}^{(n)}_\rho \qquad \left(B^{(\Delta n)} \cap \bar{X}^{(n)}_\rho = [\psi^{(n)}] \cap \bar{X}^{(n)}_\rho\right).$$

(a) is immediate from (15) since $B^{(*0)} = B^*$ and since $\overline{\psi(\cong)} = \psi(\cong)(\approx)$ $= \psi$ when $=$ does not occur in ψ.

Consider the initial step in the inductive proof of (15). Let B be a basic clopen set in $X_{\bar{\rho}}$. Then for some m and some basic$'$ m-formula ψ of $L_{\omega\omega}(\bar{\rho})$ which does not involve the equality symbol, $B = \{S: (\omega, S, 0, \ldots, m-1) \models \psi\}$. We know (e.g. from [15]) that $B^{(\Delta n)} = [(\exists^{\neq} \mathbf{v}_n \cdots \mathbf{v}_m)(\psi)^{(n)}]$. By (15),

$$[(\exists^{\neq} \mathbf{v}_n \cdots \mathbf{v}_m)(\psi)^{(n)}] \cap \bar{X}^{(n)}_\rho = [(\exists \mathbf{v}_n \cdots \mathbf{v}_m)(\psi)^{(n)}] \cap \bar{X}^{(n)}_\rho.$$

The remaining steps are similar. At each stage we carry the additional hypothesis that the formulas defined previously do not contain $=$; we use the argument from [15] to construct a new formula; then we use (14) to eliminate the equality symbol from that new formula.

This completes the proof of 2.1(a). As in [15], (b) is immediate from (13), (a), and the fact that B^* is an invariantization of B. $\square$

The syntactical objects corresponding to alternated unions are alternated disjunctions. Given a sequence $\Phi = \langle \phi_\beta : \beta \leqslant \gamma \rangle$ of ρ-sentences, let

$$D(\Phi) = \bigvee \{\phi_\beta \wedge \neg \phi_{\beta+1} : \beta \in e(\gamma)\}.$$

A sequence $\langle \phi_\beta : \beta \leqslant \gamma \rangle$ of $\Pi'^0_{(\alpha)}$ sentences is suitable for $\mathfrak{D}_\gamma(\Pi'^0_{(\alpha)})$ if $\langle \mathrm{Mod}(\phi_\beta) : \beta \leqslant \gamma \rangle$ is suitable for $\mathfrak{D}_\gamma(\Pi'^0_{(\alpha)}(V_\rho))$.

$$\mathfrak{D}_\gamma(\Pi'^0_{(\alpha)}) = \{D(\Phi) : \Phi \text{ is suitable for } \mathfrak{D}_\gamma(\Pi'^0_{(\alpha)})\}.$$

$$\mathfrak{D}_{(\omega_1)}(\Pi'^0_{(\alpha)}) = \bigcup_{\alpha < \omega_1} \mathfrak{D}_\gamma(\Pi'^0_{(\alpha)}).$$

THEOREM 2.2. *Assume* $1 < \alpha < \omega_1$. *For each* $\gamma \in \omega_1$,

$$\mathrm{inv}\left(\mathfrak{D}_\gamma\left(\Pi^0_{(\alpha)}\left(\bar{X}_\rho\right)\right)\right) = \overline{\mathfrak{D}_\gamma(\Pi'^0_{(\alpha)})}\left(\bar{X}_\rho\right)$$

and

$$\mathrm{inv}\left(\Delta^0_\alpha\left(\bar{X}_\rho\right)\right) = \overline{\mathfrak{D}_{(\omega_1)}(\Pi'^0_{(\alpha)})}\left(\bar{X}_\rho\right).$$

PROOF. This follows directly from 1.4, 2.1(b) and the fact that a sequence $\langle\phi_\beta: \beta \leqslant \gamma\rangle$ is suitable for $\mathfrak{D}_\gamma(\mathbf{\Pi}'^0_\alpha)$ if and only if $\langle(\!|\phi_\beta|\!): \beta \leqslant \gamma\rangle$ is suitable for $\mathfrak{D}_\gamma(\mathbf{\Pi}^0_\alpha(\bar{X}_\rho))$. □

Now we can prove the full $\mathbf{\Pi}'^0_\alpha$ separation theorem.

THEOREM 2.3. *Let ρ be a countable similarity type and let $\alpha \geqslant 2$. Then the collection $\mathbf{\Pi}'^0_\alpha(V_\rho)$ has the strong separation property with respect to $\mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_{(\alpha)})(V_\rho)$.*

PROOF. Let $\mathrm{Mod}(\theta_1)$, $\mathrm{Mod}(\theta_2)$ be disjoint $\mathbf{\Pi}'^0_\alpha$ classes. Then $(\!|\theta_1|\!)$, $(\!|\theta_2|\!)$ are disjoint invariant $\mathbf{\Pi}^0_\alpha$ subsets of $X_{\bar{\rho}}$. By 1.4(c) there is a set $D \in \mathfrak{D}_{(\omega_1)}(\mathrm{inv}(\mathbf{\Pi}^0_{(\alpha)}(X_{\bar{\rho}})))$ which separates $(\!|\theta_1|\!)$ from $(\!|\theta_2|\!)$. By 2.2, $D \cap \bar{X}_\rho = (\theta)$ for some $\phi \in \mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_{(\alpha)})$.

Clearly $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$ over countable models and by the Löwenheim-Skolem theorem for $L_{\omega_1\omega}(\rho)$, $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$ over all models. □

REMARKS. I. We have lost no generality in assuming that our similarity type contained only relation symbols. Suppose ρ is a type which includes some operation symbols. Let $\hat{\rho}$ be the result of replacing each n-ary operation symbol with an $n+1$-ary relation symbol. The canonical embedding maps V_ρ to a $\mathbf{\Pi}'^0_2$ subclass of $V_{\hat{\rho}}$ and the $\mathbf{\Pi}'^0_\alpha$ separation theorem for V_ρ is easily derived from 2.3 applied to $V_{\hat{\rho}}$.

The countability assumption on ρ seems to be essential for the separation theorem. It is apparent that the definability results 2.1 and the first part of 2.2 extend to uncountable similarity types in full analogy with the results in [15].

II. As mentioned in the introduction, we can apply an approximation theorem of J. Keisler to derive the $\mathbf{\forall}^0_n$ separation theorem from 2.3. For present purposes it suffices to define for $n \in \omega$, $\mathbf{\forall}^0_n = L_{\omega\omega}(\rho) \cap \mathbf{\Pi}'^0_n$. $\mathbf{B}^0_n$ is the closure of $\mathbf{\forall}^0_n$ under negation and finite disjunction. Given a formula $\phi \in L_{\omega_1\omega}(\rho)$, let $\phi\neg$ be the equivalent negation-normal formula which is obtained by repeated application of the infinitary de Morgan rules. Keisler's approximations as defined in [7] have two important features:

(16) Suppose $\theta_1, \theta_2 \in L_{\omega\omega}(\rho)$, ϕ is in negation-normal form, and $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$. Then there is an approximation $\sigma \in \mathfrak{A}^c(\phi)$ [7] such that $\mathrm{Mod}(\sigma)$ also separates $\mathrm{Mod}(\theta_1)$ from $\mathrm{Mod}(\theta_2)$.

(17) If $\phi \in \mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_n)$ and $\psi \in \mathfrak{A}^c(\phi\neg)$, then $\mathrm{Mod}(\psi) \in B^0_n(V_\rho)$.

Combining (16) and (17) with 2.3 we obtain

(18) (Shoenfield) For $n \geqslant 2$ the collection $\mathbf{\forall}^0_n(V_\rho)$ has the strong separation property with respect to $\mathbf{B}^0_{n-1}(V_\rho)$.

To verify (18), let θ_1, θ_2 be mutually inconsistent members of $\mathbf{\forall}^0_n(\rho)$. By 1.3, there exists $\phi \in \mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_{n-1}(\rho))$ such that $\mathrm{Mod}(\phi)$ separates $\mathrm{Mod}(\theta_1)$ from

Mod(θ_2). By (15), the same is true of some $\sigma \in \mathcal{Q}^c(\phi^{\urcorner})$. By (17), Mod($\sigma$) $\in$ $\mathbf{B}^0_n(V_\rho)$.

In his dissertation **[12]**, Myers proved a separation theorem for multiplicative classes in the $L_{\omega_1\omega}$ hierarchy based on quantifier depth (without regard to infinite conjunction and disjunction). Myers' result also yields Shoenfield's via the approximation theory, but it is much less natural topologically. We do not know a topological theorem about logic spaces from which Myers' result can be obtained.

III. The $\mathbf{\Pi}'^0_\alpha$ separation theorem for successor α, $\alpha \geqslant 2$, can be reduced to the case $\alpha = 2$ by the following method. The method seems to be essential for the effective theorem of §3. It shows that the *-transform can be avoided in deriving 2.3 for successor α (though apparently not for limit α, nor for definability results such as 2.2).

Let ρ be countable and suppose K_0, $K_1 \in \mathbf{\Pi}'^0_{\beta+1}(V_\rho)$ are disjoint, $\beta \geqslant 2$. For $i = 0, 1$ choose

$$\theta_i = \bigwedge_n \forall \mathbf{v}_0 \cdots \mathbf{v}_{k^i_n} \bigvee_m \exists \mathbf{v}_{k^i_n+1} \cdots \mathbf{v}_{k^i_{nm}} \theta^i_{nm}$$

such that each $\theta^i_{nm} \in \mathbf{\Pi}'^0_{(\beta)}(\rho)$, $K_i = \mathrm{Mod}(\theta_i)$. Let L be the smallest fragment of $L_{\omega_1\omega}(\rho)$ which contains each θ^i_{nm}. Let $\rho^\# = \rho^{\#L}$ be the similarity type which contains an n-ary predicate $\mathbf{R}_{\phi,n}$ for each $n \in \omega$ and each n-formula $\phi \in L$. Given $\mathfrak{A} \in V_\rho$, σ a sentence of $L_{\omega_1\omega}(\rho)$, let $\mathfrak{A}^\# \in V_{\rho^\#}$ be the canonical expansion of $\mathfrak{A}$, and let $\mathrm{Mod}^\#(\sigma) = \{\mathfrak{A}^\#: \mathfrak{A} \models \sigma\}$.

Let $V^\#_\rho = \{\mathfrak{A}^\#: \mathfrak{A} \in V_\rho\}$. Note that:

(19) $\mathrm{Mod}^\#(\theta_i) \in \mathbf{\Pi}'^0_2(V^\#_\rho)$, $i = 1, 2$.

(20) If $\phi \in \mathbf{\Pi}'^0_1(\rho^\#)$, then $\mathrm{Mod}(\phi) \cap V^\#_\rho = \mathrm{Mod}^\#(\psi)$ for some $\psi \in \mathbf{\Pi}'^0_\beta(\rho)$.

By (19) and the $\mathbf{\Pi}'^0_2$ separation theorem for $\rho^\#$, there exists $\phi \in \mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_1(\rho^\#))$ such that Mod(ϕ) separates $\mathrm{Mod}^\#(\theta_1)$ from $\mathrm{Mod}^\#(\theta_0)$. By (20), $\mathrm{Mod}(\phi) \cap V^\#_\rho = \mathrm{Mod}^\#(\psi)$ for some $\psi \in \mathfrak{D}_{(\omega_1)}(\mathbf{\Pi}'^0_\beta(\rho))$. Then Mod($\phi$) separates Mod($\theta_1$) from Mod($\theta_0$).

After proving 2.3 we learned from Myers that (at least for successor α and over infinite models) it was an unpublished result of G. E. Reyes. He apparently derived the case $\alpha = 2$ from Hausdorff's proof and the fact that the closure of any invariant subset of X_ρ is closed′, and then translated the result to other successor α using Skolem predicates (presumably by the preceding argument).

IV. The $\mathbf{\Pi}^0_\alpha$ (1st) separation principle is often presented as a corollary to the $\mathbf{\Sigma}^0_\alpha$ reduction principle. It is natural to ask whether this reduction principle has an invariant version. The following example shows that invariant reduction fails in the canonical logic action (and hence also in logic).

PROPOSITION 2.4. *Let ρ consist of a single binary relation and let $I = I_\rho$ be the canonical equivalence on $X_\rho = 2^{\omega \times \omega}$. Let $A_0 = \{R$: $(\exists n)(\forall m)(R(n, m) =$*

1)}, $A_1 = \{R: (\exists m)(\forall n)(R(n, m) = 1)\}$. *Then there is no pair of I-invariant* Σ^0_2 *sets which reduces* (A_0, A_1).

PROOF. Choose R_0 so that (ω, R_0) is a dense linear order with left and right endpoints (i.e. an order of type $1 + \eta + 1$). Suppose B is an invariant $\mathbf{\Pi}^0_2$ set which contains R_0. Then $B = [\theta]$ for some $\mathbf{\Pi}'^0_2$ sentence θ. Since $\mathbf{\Pi}'^0_2$ classes are closed under unions of chains (cf. Weinstein **[16]**), B has members R_1 and R_2 which define orders of type $\eta + 1$ and $1 + \eta$, respectively; hence B cannot include either $\sim A_0 \cap A_1$ or $\sim A_1 \cap A_0$.

Suppose (B_0, B_1) is a pair of Σ^0_2 sets which reduces (A_0, A_1). Then, for $i = 0$ or 1, $R_0 \in \sim B_i$ and $\sim A_i \cap A_{1-i} \subseteq \sim B_i$. By the argument of the preceding paragraph, B_i is not invariant. □

3. Remarks on orbits. Let ρ be a fixed countable similarity type.

Given $n \in \omega$ and an n-formula $\phi = \phi(\mathrm{v}_0 \cdots \mathrm{v}_{n-1}) \in L_{\omega_1\omega}(\rho)$, define

$$\ulcorner\phi\urcorner = \{R \in X_\rho: (\omega, R, 0, \ldots, n-1) \models \phi\}.$$

Let L be a countable fragment of $L_{\omega_1\omega}(\rho)$, and let X^L be the topological space formed on the underlying set of X_ρ by taking $\{\ulcorner\phi\urcorner: \phi \in L\}$ as a basis.

Given $R \in X^L$ we continue to identify R with (ω, R). Th(R) is the L-theory of (ω, R) and $[R]$ is the orbit of R under the canonical action. $[R]$ is Borel, and, in general, there will be orbits of arbitrarily high Borel rank. In **[5]** M. Benda proved a result relating a model theoretic condition on R to the topological complexity of $[R]$ in $X^{L_{\omega\omega}}$; viz.

(21) *If R is saturated and* Th(R) *is not ω-categorical, then* $[R]$ *is not* Σ^0_2 *in* $X^{L_{\omega\omega}}$.

Topological questions about orbits in $X^{L_{\omega\omega}}$ were also considered briefly by Suzuki **[14]**.

In this section we will obtain further results of this kind. The invariant $\mathbf{\Pi}^0_\alpha$ separation theorem will be an important tool. Both 3.2 and 3.5 will improve (21).

Let $\rho^{\#L}$ be the similarity type with a Skolem predicate $\mathbf{P}_\phi$ for each formula $\phi \in L$. Then the canonical embedding $J: R \mapsto R^{\#L}$ of X^L into $X_{\rho^{\#L}}$ defines a homeomorphism of X^L with an invariant $\mathbf{\Pi}^0_2$ subset of $X_{\rho^{\#L}}$. It follows that X^L is Polish and $(\omega!, X^L, J_\rho)$ is a Polish action. Moreover, since the canonical embedding commutes with the canonical actions on X_ρ and $X_{\rho^{\#L}}$, Vaught's result (12) can be translated into a definability result for X^L. The definition of the classes L-Σ'^0_α, L-$\mathbf{\Pi}'^0_\alpha$ (read "Σ'^0_α-over-L", etc.), is obtained from the definition of Σ'^0_α, $\mathbf{\Pi}'^0_\alpha$, by replacing the initial class $\mathbf{\Pi}'^0_0$ by L and retaining the inductive clauses as stated.

We have

(22) For $\alpha \geqslant 1$, invariant $\Sigma^0_\alpha(X^L) = L$-$\Sigma'^0_\alpha(X_\rho)$.

PROOF. Inclusion from right to left is trivial. To go from left to right assume

$B \in \text{inv}(\Sigma^0_\alpha(X^L))$; then $J(B) \in \text{inv}(\Sigma^0_\alpha(J(X^L)))$. By (12), $J(B) = [\phi] \cap J(X^L)$ for some $\theta \in \Sigma'^0_\alpha(\rho^{\# L})$. Let ψ be the result of replacing each atomic subformula of θ by the corresponding formula of L. Then ψ is L-Σ'^0_α and $B = [\psi]$. □

Our first result provides the second half of the "inverse" to Suzuki's observation [**14**, Theorem 2] that the orbit of a prime model R is a comeager Π^0_2 subset of $[\bigwedge \text{Th}(R)] \subseteq X^L$. ([**14**, Theorem 3] is the first half. Suzuki worked with $L = L_{\omega\omega}$ but his arguments work in the general context considered here.)

PROPOSITION 3.1. *If* $[R] \in \Pi^0_2(X^L)$, *then* (ω, R) *is L-atomic (every finite sequence from* ω *realizes a principal L-type in* (ω, R)).

PROOF. This result follows easily from the Baire Category Theorem and Suzuki's theorem [**14**, Theorem 3] that a model with a comeager orbit is atomic. The following direct proof, however, introduces some ideas which are essential for our further results.

Suppose $[R] \in \Pi^0_2$. Since L is closed under quantification, it follows from (22) that we can write $[R] = [\bigwedge_n \forall v_0 \cdots v_{n-1} \bigvee_m \phi_{nm}]$, where each ϕ_{nm} is an n-formula of L. Let $\Delta_n = \{\neg\phi_{nm}: m \in \omega\}$; then $[R] = \{S: S$ omits each type $\Delta_n, n \in \omega\}$. If R realized a nonprincipal type Σ, we could find S which omits $\{\Sigma\} \cup \{\Delta_n: n \in \omega\}$. But then $S \in [R]$ and $S \not\cong R$, a contradiction. □

Note that $[\bigwedge \text{Th}(R)]$ is the closure of $[R]$ in X^L, hence

(23) $[R]$ is closed if and only if $\text{Th}(R)$ is ω-categorical.

In view of the intrinsic invariance of the Borel classes (cf. Kuratowski [**9**, §35]), for every α, $[R]$ is a Σ^0_α (or Π^0_α) subset of X^L if and only if $[R]$ is a Σ^0_α (Π^0_α) subset of $[\bigwedge \text{Th}(R)]$. In view of this fact, and of (23), we lose no information by studying the complexity of orbits relative to $[\bigwedge T]$ where T is a complete L-theory which is not ω-categorical. It should also be noted that in view of (22), all results of this section could be stated in terms of $L_{\omega_1\omega}$ definability and without explicit reference to any topological space.

For the remainder of §3 *we assume T is a fixed, complete not* ω*-categorical theory of* $L_{\omega\omega}$ *and* $X = X^T = [\bigwedge T]$ *with the relative topology as a subspace of* $X^{L_{\omega\omega}}$.

X is exactly the space **S** studied in [**5**].

Following Benda [**5**] we say R is *full* (weakly saturated) if every elementary type over T is realized in R. An elementary type Δ is *powerful* if every model of T which realizes Δ is full.

THEOREM 3.2. *No orbit is* Σ^0_2.

PROOF. Suppose $[R] \in \Sigma^0_2(X)$; then by (22), $[R] = [\bigvee_n \exists v_0 \cdots v_{n-1} \bigwedge_m \phi_{nm}]$ for some collection $\{\phi_{nm}: n, m \in \omega\}$ such that each ϕ_{nm} is an

n-formula of $L_{\omega\omega}$. Since $[R]$ is minimal invariant, there is some n_0 such that $[R] = [\exists \mathbf{v}_0 \cdots \mathbf{v}_{n_0-1} \bigwedge_m \phi_{n_0 m}]$ i.e. $[R] = \{S: \Delta \text{ is realized in } S\}$ where Δ is the n_0-type $\{\phi_{n_0 m}: m \in \omega\}$.

If R is not full, let Σ be a complete type over T which is omitted by R, and let S realize both Δ and Σ. Then $S \in [R]$ and $S \not\cong R$, a contradiction.

If R is full, then Δ is powerful and, since T is not ω-categorical, there are both saturated and nonsaturated models which realize Δ, again contradicting the fact that $[R]$ is an orbit. □

LEMMA 3.3. *If R is full and G is an invariant $\mathbf{\Pi}_2^0$ set which contains R, then $G = X$.*

PROOF. It suffices to prove the lemma for $G = [\forall \mathbf{v}_0 \cdots \mathbf{v}_{n-1} \bigvee_m \phi_m]$, each $\phi_m \in L_{\omega\omega}$, since every invariant $\mathbf{\Pi}_2^0$ set is an intersection of sets of this form. Let $\Delta = \{\neg\phi_m: m \in \omega\}$. Then $G = \{S: S \text{ omits } \Delta\}$. Since R is full, every model of T omits Δ. □

THEOREM 3.4. *No full model has a $\mathbf{\Delta}_3^0$ orbit.*

PROOF. Suppose R is full and $[R] \in \mathbf{\Delta}_3^0$. Then by 1.4, $[R] \in \mathfrak{D}_{(\omega_1)}(\text{inv}(\mathbf{\Pi}_2^0(X)))$, and since $[R]$ is minimal invariant, $[R] = G_1 \sim G_2$ for some invariant $\mathbf{\Pi}_2^0$ sets G_1, G_2. By 3.3, $G_1 = X$ and $[R] = \sim G_2$, contradicting 3.2. □

COROLLARY 3.5. (i) *If R is saturated, then $[R] \in \mathbf{\Pi}_3^0 \sim \mathbf{\Sigma}_3^0$.*

(ii) *Let $\mathbf{c}_0, \ldots, \mathbf{c}_n$ be constant symbols not in ρ and let $p' = \rho \cup \{\mathbf{c}_0, \ldots, \mathbf{c}_n\}$. If Δ is a powerful $n+1$-type over T and $(\omega, S, i_0, \ldots, i_n)$ is a prime model of a complete extension of*

$$\Delta\begin{pmatrix} \mathbf{v}_0 \cdots \mathbf{v}_n \\ \mathbf{c}_0 \cdots \mathbf{c}_n \end{pmatrix} \subseteq L_{\omega\omega}(\rho'),$$

then $[S] \in \mathbf{\Sigma}_3^0 \sim \mathbf{\Pi}_3^0$.

PROOF. It is easy (see [5]) to see that R, S belong to $\mathbf{\Sigma}_3^0$, $\mathbf{\Pi}_3^0$, respectively. The conclusion then follows by 3.4. □

We have a partial converse to 3.5(i).

THEOREM 3.6. *Assume R is full and $[R] \in \mathbf{\Pi}_3^0$. Then R is saturated.*

PROOF. Suppose R is not saturated. Since R is full, T has a countable saturated model S. Then $[R]$ and $[S]$ are disjoint minimal invariant $\mathbf{\Pi}_3^0$ sets. It follows from the invariant $\mathbf{\Pi}_3^0$ separation theorem that there are invariant $\mathbf{\Pi}_2^0$ sets G_1, G_2 such that $[R] \subseteq G_1 \sim G_2 \subseteq \sim [S]$. Since R is full, it follows from 3.3 that $G_1 = X$. Then $[S] \subseteq G_2$, and since S is full, $G_2 = X$ and $[R] = \varnothing$, a contradiction. □

The invariant $\mathbf{\Pi}_\alpha^0$ separation principle appears to be a useful tool for attacking general classification problems in descriptive set theory. For example, consider the following proof of one of the first results in the subject (cf. Addison [3] or Lusin [10]).

(24) (Baire 1906) The set $A = \{R \in 2^{\omega\times\omega}: R$ defines a function $f_R: \omega \to \omega$ & $(\forall n)(f_R^{-1}(\{n\})$ is finite)$\}$ belongs to $\Pi^0_3 \sim \Sigma^0_3$.

PROOF. A is obviously invariant Π^0_3. If A were Σ^0_3 then A would be an alternated union of invariant Π^0_2 sets. Such sets cannot separate structures which satisfy the same Π'^0_2 sentences (i.e. which realize the same types of $\forall^0_1$ formulas). It is easy to show that A can.

Consider, for example, the functions f_0, f_1 defined as follows:

(i) If $j = p^n$ where p is the ith odd prime and $1 \leqslant n \leqslant i$, then $f_1(j) = p$; otherwise $f_1(j) = j$.

(ii) If j is odd, then $f_2(j) = f_1(j)$; if j is even, then $f_2(j) = 0$.

Let R_i be the characteristic function of f_i, $i = 1, 2$. Then $R_1 \in A$, $R_2 \notin A$, and it is a straightforward exercise to show that (ω, R_1), (ω, R_2) realize the same types of $\forall^0_1$ formulas. □

4. Π'^0_α separation and the problem of effectiveness. The main result of this section, (4.2), is an "admissible" version of the $\Pi^0_\alpha(X_\rho)$ separation theorem for $\rho \in \mathrm{HC}$ and $\alpha \geqslant 2$, a successor ordinal. $\mathrm{HC} = \{x: \overline{\overline{\mathrm{TC}(x)}} \leqslant \omega\}$ is the collection of hereditarily countable sets. $\mathrm{TC}(x)$ is the transitive closure of x. Since the construction used in 2.3 is highly effective, we will obtain a corresponding Π'^0_α separation theorem for admissible languages as a corollary.

We continue to assume for convenience that ρ is a set of relation symbols. We further assume that $\rho \in \mathrm{HC}$ and that the syntax of $L_{\omega_1\omega}$ is formalized in a standard fashion so that $L_{\omega_1\omega}(\rho) \subseteq \mathrm{HC}$ and the usual syntactical operations (substitution, collection of subformulas, etc.) are primitive recursive, (cf. Barwise [4]). For definiteness we specify that a relation symbol is a triple $\mathbf{R} = (1, s, n)$, where s is arbitrary and $n = n(\mathbf{R}) \in \omega$, a constant symbol is a pair $\mathbf{s} = (2, s)$, and that the language is then constructed as in Keisler [8]. The facts about admissible sets and primitive recursive (prim) set functions which we require may be found in [8] and in Jensen and Karp [6].

Let $C = \{(2, n): n \in \omega\}$. Borel subsets of X_ρ are naturally named by variable-free (propositional) sentences of $L_{\omega_1\omega}(\rho \cup C)$. Such sentences will be called ρ-names. When discussing ρ-names we will omit "primes" and refer to basic ρ-names, Π^0_α-ρ-names, etc. A ρ-name θ names the set $[\theta] = \{S: (\omega, S, 0, 1, \dots) \models \theta\}$. It is apparent that $B \subseteq X_\rho$ is Π^0_α if and only if $B = [\theta]$ for some Π^0_α-ρ-name θ.

$\Pi^0_\alpha(\rho)$ is the set of Π^0_α-ρ-names. $\Pi^0_{(\alpha)}(\rho) = \bigcup\{\Pi^0_\beta(\rho): \beta < \alpha\}$. $\mathrm{at}(\rho)$ is the set of atomic ρ-names. For $\mathcal{Q} \subseteq \mathrm{HC}$, $\Pi^0_\alpha[\mathcal{Q}] = \Pi^0_\alpha(\rho) \cap \mathcal{Q}$. $\mathrm{at}(\rho) \cup \{\neg\phi$:

$\phi \in \text{at}(\rho)\}$ is the set of subbasic ρ-names. Given a name θ, sub(θ) is the set of subnames (subformulas) of θ.

The following lemma is an effective version of the classical method of generalized homeomorphisms (i.e. the classical method of Skolem predicates). It will be used to reduce the general case of the $\Pi^0_{\alpha+1}$ separation theorem to the case $\alpha = 1$.

LEMMA 4.1. *Let* $\mathcal{Q} \subseteq \text{HC}$ *be prim-closed,* $\omega, \rho \in \mathcal{Q}$, $1 < \alpha < \omega_1$. *Suppose* $\Theta \in \mathcal{Q}$, $\Theta \subseteq \Pi^0_{(\alpha)}(\rho)$. *Then there exist* $\rho_1, \Psi, \mathfrak{F}_0, \mathfrak{F}_1 \in \mathcal{Q}$, $g\colon X_\rho \to X_{\rho_1}$ *such that*:

(i) ρ_1 *contains only* 0-*ary relation symbols* (*i.e.*, ρ_1 *is propositional*).

(ii) Ψ *is a* Π^0_2-ρ_1-*name and g is a Borel isomorphism on* X_ρ *onto* $[\Psi]$.

(iii) $\mathfrak{F}_0\colon \text{at}(\rho_1) \to \Sigma^0_\alpha(\rho)$, $\mathfrak{F}_1\colon \Theta \to \text{at}(\rho_1)$ *are functions such that for* $\psi \in \text{at}(\rho_1)$, $\theta \in \Theta$, $[\mathfrak{F}_0(\psi)] = g^{-1}([\psi])$, *and* $[\mathfrak{F}_1(\theta)] \cap [\Psi] = g([\theta])$.

PROOF. Let $L = \{\text{at}(\rho)\} \cup \{\text{sub}(\theta)\colon \theta \in \Theta\}$. Let ρ_1 be the similarity type with a 0-ary relation symbol $\mathbf{P}_\phi = (1, (L, \phi), 0)$ for each $\phi \in L$. Let $\Psi \in \mathcal{Q}$ be a Π^0_2-ρ_1-name for B_1 where

$$B_1 = \bigcap_{\neg\phi \in L} \left[\mathbf{P}_{\neg\phi} \leftrightarrow \neg\mathbf{P}_\phi\right] \cap \bigcap_{\bigvee\Gamma \in L} \left[\mathbf{P}_{\bigvee\Gamma} \leftrightarrow \bigvee_{\gamma\in\Gamma} \mathbf{P}_\gamma\right]$$
$$\cap \bigcap_{\bigwedge\Gamma \in L} \left[\mathbf{P}_{\bigwedge\Gamma} \leftrightarrow \bigwedge_{\gamma\in\Gamma} \mathbf{P}_\gamma\right].$$

Let $\mathfrak{L}\colon L \to \Sigma^0_\alpha(\rho)$ be a prim function such that for every $\phi \in L$, $[\mathfrak{L}(\phi)] = [\phi]$. Define $\mathfrak{F}_0 = \{(\mathbf{P}_\phi, \mathfrak{L}(\phi))\colon \phi \in L\}$, $\mathfrak{F}_1 = \{(\theta, \mathbf{P}_\theta)\colon \theta \in \Theta\}$. For $R \in X_\rho$, set $g(R)(\mathbf{P}_\phi) = 1$ if $R \in [\phi]$, 0 otherwise. It is easily checked that ρ_1, Ψ, g, $\mathfrak{F}_0$, $\mathfrak{F}_1$ have the required properties. □

Given a sequence $\phi = \langle \phi_\beta\colon \beta < \gamma\rangle$ of ρ-names, let $[\phi] = \langle [\phi_\beta]\colon \beta < \gamma\rangle$. Let

$$\Pi^0_\mu[\mathcal{Q}](X_\rho) = \{[\varphi]\colon \varphi \in \Pi^0_\mu[\mathcal{Q}]\},$$

$$\mathfrak{D}_{[\mathcal{Q}]}(\Pi^0_\mu(X_\rho)) = \Big\{D[\varphi]\colon (\exists\delta \in \omega_1)\big(\varphi \in \mathcal{Q} \cap {}^{\delta+1}(\Pi^0_\mu(\rho))$$
$$\&[\varphi] \text{ is suitable for } \mathfrak{D}_\delta(\Pi^0_\mu(X_\rho))\big)\Big\}.$$

THEOREM 4.2. *Suppose* $\mathcal{Q} \subseteq \text{HC}$ *is admissible,* $\rho, \omega \in \mathcal{Q}$, $1 \leqslant \mu < \omega_1$. *Then* $\Pi^0_{\mu+1}[\mathcal{Q}](X_\rho)$ *has the strong separation property with respect to* $\mathfrak{D}_{[\mathcal{Q}]}(\Pi^0_\mu(X_\rho))$.

PROOF. We consider two cases.

Case 1. $\mu = 1$ and ρ is propositional.

Let A_0, A_1 be disjoint sets belonging to $\Pi^0_2[\mathcal{Q}](X_\rho)$.

We may assume

$$A_1 = \bigcap_{k\in K_1} \bigcup_{j\in J} [\theta_{kj}], \qquad A_2 = \bigcap_{k\in K_2} \bigcup_{j\in J} [\theta_{kj}],$$

where each θ_{kj} is a finite conjunction of subbasic names, say $\theta_{kj} = \bigwedge t_{kj}$, K_1, K_2 are disjoint elements of $\mathcal{Q}$, and the sequences $\langle \theta_{kj}: k \in K_1, j \in J \rangle$, $\langle \theta_{kj}: k \in K_2, j \in J \rangle$ belong to $\mathcal{Q}$.

Let σ, τ range over the set T of finite functions with domain included in $K_1 \cup K_2$, range included in J. (These are "partial Skolem functions".) Let s, t range over the collection Γ of finite sets of subbasic ρ-names ("partial elements of X_ρ"). Given $\sigma \in T$, let $t^\sigma = \bigcup\{t_{k\sigma(k)}: k \in \mathrm{dom}(\sigma)\}$, $\theta^\sigma = \bigwedge t^\sigma$, so $[\theta^\sigma] = \bigcap_{k \in \mathrm{dom}(\sigma)}[\theta_{k\sigma(k)}]$.

Given $\alpha \in \mathrm{On}$, let $p(\alpha) = 1$ if α is odd, 2 if α is even. Choose a set $\infty \notin \mathrm{On} \cup \mathcal{Q}$. Define a rank function Rk: $\Gamma \times T \to \mathrm{On} \cup \{\infty\}$ by the conditions:

$$\mathrm{Rk}(s, \sigma) \geq 1 \quad \text{if } [\bigwedge s] \cap [\theta^\sigma] \neq \varnothing,$$

$$\begin{aligned}\mathrm{Rk}(s, \sigma) \geq \alpha + 1 \quad &\text{if } (\forall \mathbf{P} \in \rho)(\forall k \in K_{p(\alpha)})(\exists t, \tau)\\ &(s \subseteq t \,\&\, \sigma \subseteq \tau \,\&\, (\mathbf{P} \in t \text{ or } \neg \mathbf{P} \in t)\\ &\qquad \& \, k \in \mathrm{dom}(\tau) \,\&\, \mathrm{Rk}(t, \tau) \geq \alpha),\end{aligned}$$

$$\mathrm{Rk}(s, \sigma) \geq \lambda \quad \text{if } \mathrm{Rk}(s, \sigma) \geq \beta \text{ for every } \beta < \lambda, \text{ provided } \lambda = \bigcup \lambda,$$

$$\mathrm{Rk}(s, \sigma) = \begin{cases} \text{the smallest } \alpha \text{ such that } \mathrm{Rk}(s, \sigma) \ngeq \alpha + 1 & \text{if such exists,} \\ \infty \quad \text{otherwise.} \end{cases} \tag{25}$$

Note $[\bigwedge s] \cap [\bigwedge t^\sigma] = \varnothing$ if and only if

$$(\exists \mathbf{P} \in \rho)(\exists k \in \mathrm{dom}(\sigma))(\mathbf{P}, \neg \mathbf{P} \in s \cup t_{k\sigma(k)}).$$

Thus, the relation on s, σ: "$\mathrm{Rk}(s, \sigma) \geq 1$" is definable by a Δ_0 formula in the parameters Γ, T, $\rho \in \mathcal{Q}$. It follows from the form of (25) that the relation on s, σ, α: "$\mathrm{Rk}(s, \sigma) \geq \alpha$" is primitive recursive in Γ, T, ρ, hence $\mathrm{Rk} \cap \mathcal{Q}$ is $\boldsymbol{\Delta}$-definable in $\mathcal{Q}$. We claim

$$\mathrm{Rk} \in \mathcal{Q}. \tag{28}$$ [1]

Let us defer verification of (28) and proceed. For $s \in \Gamma$ let $\check{s} = \{\neg \mathbf{P}: \mathbf{P} \in s\} \cup \{\mathbf{P}: \neg \mathbf{P} \in s\}$. Let $\delta = \mathrm{range(Rk)}$. For $\eta \leq \delta$, define

$$r_\eta = \{\bigvee(\check{t}^\sigma \cup \check{s}): (s, \sigma) \in \Gamma \times T \,\&\, \mathrm{Rk}(s, \sigma) < \eta\},$$

$\varphi_\eta = \bigwedge r_\eta$, so

$$[\varphi_\eta] = \bigcap_{\mathrm{Rk}(s,\sigma) < \eta} \sim([\bigwedge s] \cap [\theta^\sigma]).$$

Let $\varphi = \langle \varphi_\eta: \eta \leq \delta \rangle$. φ is primitive recursive in parameters from $\mathcal{Q}$, $\mathrm{dom}(\varphi) = \delta + 1 \in \mathcal{Q}$, hence $\varphi \in \mathcal{Q}$. It is easy to check that $[\varphi]$ is suitable for

[1] Lemmas in the text are numbered in logical order.

$\mathfrak{D}_\delta(\Pi_1^0(X_\rho))$. We claim

$$A_1 \subseteq D([\varphi]) \subseteq \sim A_2. \tag{30}$$

To establish (30), suppose $x \in X_\rho$. Let

$$\eta_x = \min\{\mathrm{Rk}(s, \sigma): x \in [\bigwedge s] \cap [\theta^\sigma]\},$$

and choose s, σ such that $x \in [\bigwedge s] \cap [\theta^\sigma]$ and $\mathrm{Rk}(s, \sigma) = \eta_x$. Note that $\eta_x \geqslant 1$ and $x \in [\varphi_{\eta_x}] \sim [\varphi_{\eta_x+1}]$. Thus, it suffices to show:

$$x \in A_i \Rightarrow p(\eta_x) \neq i \quad \text{for } i = 1, 2. \tag{29}$$

Since $\mathrm{Rk}(s, \sigma) < \eta_x + 1$, there is some $\mathbf{P} \in \rho$, $k \in K_{p(\eta_x)}$ such that for any $t \in \Gamma, j \in J$,

$$s \subseteq t \;\&\; (\mathbf{P} \in t \text{ or } \neg\mathbf{P} \in t) \Rightarrow \mathrm{Rk}(t, \sigma \cup \{(k, j)\}) < \eta_x.$$

In particular, if

$$t = \begin{cases} s \cup \{\mathbf{P}\} & \text{if } x \in [\mathbf{P}], \\ s \cup \{\neg\mathbf{P}\} & \text{otherwise,} \end{cases}$$

then $\mathrm{Rk}(t, \sigma \cup \{(k, j)\}) < \eta_x$ for any j. By the minimality of η_x it follows that

$$x \notin \bigcup_{j \in J} [\theta^{\sigma \cup \{(k,j)\}}] = [\theta^\sigma] \cap \bigcup_{j \in J} [\theta_{kj}].$$

Since $x \in [\theta^\sigma]$ we obtain $x \notin \bigcup_{j \in J}[\theta_{kj}]$, so $x \notin A_{p(\eta_x)}$. This proves (29) and, hence, (30).

It remains to establish (28). We first prove

$$\mathrm{Image(Rk)} \subseteq \mathrm{On}. \tag{27}$$

Since $\Gamma \times T$ is countable, there exists an odd ordinal $\delta < \omega_1$ such that

$$(\forall s, \sigma)(\mathrm{Rk}(s, \sigma) \geqslant \delta \Rightarrow \mathrm{Rk}(s, \sigma) \geqslant \delta + 2).$$

Let $\mathcal{P}: \omega \to \rho$, $g_1: \omega \to K_1$, $g_2: \omega \to K_2$ be surjections. If $\mathrm{Rk}(\varnothing, \varnothing) \geqslant \delta$, then $\mathrm{Rk}(\varnothing, \varnothing) \geqslant \delta + 1$, and for some t, τ,

$$(\mathcal{P}(0) \in t \text{ or } \neg\mathcal{P}(0) \in t) \;\&\; g_1(0) \in \mathrm{dom}(\tau) \;\&\; \mathrm{Rk}(t, \tau) \geqslant \delta.$$

Then $\mathrm{Rk}(t, \tau) \geqslant \delta + 2$, so there exist $s_0 \supseteq t$, $\sigma_0 \supseteq \tau$ such that $g_2(0) \in \mathrm{dom}(\sigma_0) \;\&\; \mathrm{Rk}(s_0, \sigma_0) \geqslant \delta$. Proceeding inductively, we may define s_n, σ_n for each $n \in \omega$ such that

$$(\forall n \in \omega)(\forall m \leqslant n)[s_m \subseteq s_n \;\&\; \sigma_m \subseteq \sigma_n \;\&\; (\mathcal{P}(n) \in s_n \text{ or } \neg\mathcal{P}(n) \in s_n)$$
$$\;\&\; g_1(n), g_2(n) \in \mathrm{dom}(\sigma_n) \;\&\; \mathrm{Rk}(s_n, \sigma_n) \geqslant \delta]. \tag{26}$$

Let x be the unique member of $\bigcap_{n \in \omega}[\bigwedge s_n]$ and let $h = \bigcup_n \sigma_n$. Note that h is a function on $K_1 \cup K_2$ to J. If $x \notin [\theta^{\sigma_n}]$, then since $\sim[\theta^{\sigma_n}]$ is open, there exists $m > n$ such that

$$[\bigwedge s_m] \subseteq \sim[\theta^{\sigma_n}] \subseteq \sim[\theta^{\sigma_m}]$$

and, hence, $\mathrm{Rk}(s_m, \sigma_m) = 0$. This contradicts (26) and shows that

$$x \in \bigcap_n [\theta^{\sigma_n}] = \bigcap_{k \in K_1 \cup K_2} [\theta_{kh(k)}].$$

This implies that $x \in A_1 \cap A_2$, a second contradiction, which proves $\mathrm{Rk}(\varnothing, \varnothing) < \delta$ and (27) follows.

If $\mathrm{Rk}(\varnothing, \varnothing) \notin \mathcal{A}$, then for some (s, σ), $\mathrm{Rk}(s, \sigma) = \omega_1 \cap \mathcal{A}$ and

$$(\mathcal{A}, \in) \models (\forall (t, \tau) \in \Gamma \times T)$$
$$\big[s \subseteq t \,\&\, \sigma \subseteq \tau \to (\exists \alpha)(\alpha \in \mathrm{On} \,\&\, \mathrm{Rk}(t, \tau) < \alpha)\big].$$

Applying Σ-reflection, we obtain $\omega_1 \cap \mathcal{A} \in \mathcal{A}$, a contradiction, which establishes (28) and completes the proof of Case 1.

Case 2. $\mu \geqslant 1$, ρ arbitrary.

Let A_1, A_2 be disjoint elements of $\Pi^0_{\mu+1}[\mathcal{A}](X_\rho)$ and suppose

$$A_i = \bigcap_{k \in K_i} \bigcup_{j \in J} [\theta_{kj}] \quad \text{for } i = 1, 2,$$

where K_1, K_2 are disjoint elements of $\mathcal{A}$, $\langle \theta_{kj} : k \in K_1 \cup K_2, j \in J \rangle \in \mathcal{A}$, and each $\theta_{kj} \in \Pi^0_{(\mu)}(\rho)$. Let $\Theta = \{\theta_{kj} : k \in K_1 \cup K_2, j \in J\}$ and choose $\rho_1, \psi, \mathcal{F}_0, \mathcal{F}_1 \in \mathcal{A}$ as given by 4.1. Let

$$B_i = [\psi] \cap \bigcap_{k \in K_i} \bigcup_{j \in J} [\mathcal{F}_1(\theta_{kj})], \qquad i = 1, 2.$$

Then B_1, B_2 are disjoint elements of $\Pi^0_2[\mathcal{A}](X_{\rho_1})$. Applying the result of Case 1, let $\varphi' = \langle \varphi'_\eta : \eta \leqslant \delta \rangle \in \mathcal{A}$ be a sequence of Π^0_1-ρ_1-names such that $B_1 \subseteq D([\varphi']) \subseteq B_2$. For $\beta \leqslant \delta$ let φ_β be the result of replacing in φ'_β each $\mathbf{P} \in \rho_1$ by $\mathcal{F}_0(\mathbf{P})$. Let $\varphi = \langle \varphi_\beta : \beta \leqslant \delta \rangle$. As in Remark III, §2, it is easily checked that $D[\varphi] \in \mathcal{D}_{[\mathcal{A}]}(\Pi^0_\mu(X_\rho))$ and $A_1 \subseteq D[\varphi] \subseteq A_2$. This completes the proof of 4.2. □

Let $\Pi'^0_\mu[\mathcal{A}] = \Pi'^0_\mu(\rho) \cap \mathcal{A}$ and let $\mathcal{D}_{[\mathcal{A}]}(\Pi'^0_\mu) = \mathcal{D}_{(\omega_1)}(\Pi'^0_\mu) \cap \mathcal{A}$.

Corollary 4.3. *Suppose* $\mathcal{A} \subseteq \mathrm{HC}$ *is admissible,* $\rho, \omega \in \mathcal{A}$, $1 \leqslant \mu < \omega_1$. *Then* $\Pi'^0_{\mu+1}[\mathcal{A}](V_\rho)$ *has the strong separation property with respect to* $\mathcal{D}_{[\mathcal{A}]}(\Pi'^0_\mu)(V_\rho)$.

Proof. As in Vaught [15] the proof of 2.1(a) is uniform and establishes

(31) There is a function $\theta \mapsto \langle \theta^{(*n)} : n \in \omega \rangle$ which is primitive recursive in parameters ω, ρ such that if θ is a Π^0_α-$\bar{\rho}$-name, then for every n, $\theta^{(*n)} \in \Pi'^0_\alpha(\rho)$ and $[\theta]^{(*n)} \cap \bar{X}^{(n)}_\rho = (\!|\theta^{(*n)(n)}|\!)$.

Using (31) our proof of 2.3 is easily made effective, giving 4.3 as a consequence of 4.2. □

Remarks. V. The usual "lightface" descriptive set theory concerns admissible sets of the form $x^+ = L_{\omega_1^x}[x]$ for $x \in 2^\omega$. Such sets $\mathcal{A}$ are *locally* *countable*, containing a map of ω onto x for each $x \in \mathcal{Q}$. If $\mathcal{Q}$ is locally countable, $\omega \in \mathcal{Q}$ and $\mathcal{Q}$ is prim-closed, then the standard proof of Σ^0_μ-reduction shows that $\Sigma^0_\mu[\mathcal{Q}](X_\rho)$ has the reduction property and, hence, $\Pi^0_\mu[\mathcal{Q}](X_\rho)$ has the weak first separation property ($\mu > 1$, $\rho \in \mathcal{Q}$). It seems doubtful that $\Sigma^0_\mu[\mathcal{Q}]$-reduction holds when $\mathcal{Q}$ is not locally countable.

VI. For $\mathcal{Q} = x^+$, $x \in 2^\omega$, it is known that $B \subseteq 2^\omega$ is $\Pi^0_\alpha[\mathcal{Q}]$ if and only if B is Π^0_α-in-y (in the sense of classical recursion theory) for some parameter y which is hyperarithmetic in x. Before we obtained 4.2, Richard Haas (unpublished) showed that $\Pi^0_2(2^\omega)$ has the strong separation property with respect to the class of recursive alternated unions of Π^0_1 sets. If his result can be shown to relativize to arbitrary parameters or to extend to higher levels of the hyperarithmetical hierarchy it will improve the result one obtains from 4.2 in these cases by avoiding the introduction of hyperarithmetic parameters.

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